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Three dimensional SCFT from M2 branes at conifold singularities

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Abstract: Recently, it was conjectured that parallel branes at conical singularities are related to string/M theory on $AdS \times X$ where X is an Einstein manifold. In this paper we consider coincident M2 branes near a conifold singularity when M theory is compactified on $AdS_4 \times Q^{1,1,1}$, where $Q^{1,1,1} = (\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2))/(\mathrm{U}(1) \times \mathrm{U}(1))$ is a seven dimensional Sasaki-Einstein manifold. We argue that M theory on $AdS_4 \times Q^{1,1,1}$ is dual to a three dimensional superconformal field theory. We use the fact that the three dimensional self-mirror duality is preserved by exact marginal operators, as observed by Strassler.

Keywords: M-Theory, Supersymmetry and Duality, Differential and Algebraic Geometry.

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Contents

1. Introduction

Recently, Maldacena argued that type IIB string theory on $AdS_5 \times \mathbf{S^5}$ is equivalent to $\mathcal{N} = 4$ supersymmetric SU(N) gauge theory in four dimensions [1] in the 't Hooft large N limit. The correspondence was made more precise in [2, 3] where it was stated that there is a 1-1 correspondence between the Green's functions in type IIB string theory on $AdS_5 \times \mathbf{S^5}$ and the correlation functions of gauge invariant operators of the $\mathcal{N} = 4$ supersymmetric field theory.

Two other conjectures were made in [1] relating M theory compactified on $AdS_4 \times \mathbf{S}^7$ or $AdS_7 \times \mathbf{S}^4$ to $\mathcal{N} = 8$ three dimensional field theory or (0, 2) six dimensional field theory respectively. The correspondence between the Green's functions in M theory and the correlation functions of the field theory has been studied in [4, 5, 6, 7]. Maldacena's conjectures have been generalized to theories with orbifolds [8, 9, 10, 11, 12, 13, 14, 15, 16] and to F theory [18, 19, 20].

One interesting generalization is to consider this duality for $AdS_5 \times X_5$ where X_5 is a five-dimensional Einstein manifold [21, 23, 29, 30, 32]. In general, the four-dimensional superconformal field theory on these backgrounds is different from $\mathcal{N} = 4$, SU(N) gauge theory and has fewer supersymmetries than the case of $AdS_5 \times \mathbf{S}^5$. In [21] Klebanov and Witten presented the duality between the field theory and type IIB string theory compactified on $AdS_5 \times X_5$ where X_5 is the homogeneous space $T^{1,1} = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathrm{U}(1)$. The theory on the world-volume of D3 branes is $\mathcal{N} = 1$ superconformal field theory in four dimensions. They also suggested the possibility of extending their results to M theory. In [32], Morrison and Plesser investigated M theory or type IIB theory compactified on $AdS_{p+2} \times H^{D-p-2}$ where the H^{D-p-2} are Einstein manifolds obtained as the horizons of the Gorenstein canonical singularities.

In a similar fashion to [21], we are going to consider a specific geometry $AdS_4 \times X_7$, where X_7 is a seven-dimensional compact Einstein manifold given by $Q^{1,1,1} = (\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2))/(\mathrm{U}(1) \times \mathrm{U}(1))$. Historically, $Q^{p,q,r}$ manifolds appeared in the classification of all Kaluza - Klein coset G/H seven-manifolds done by Castellani, Romans and Warner in [24]. The $Q^{p,q,r}$ manifolds were also constructed and their supersymmetries were derived by D'Auria, Fré and Van Nieuwenhuizen in [27] (see also [25, 26]). The use of G/H Einstein manifolds to generate M2–brane solutions in connection with a conformal field theory interpretation has been discussed recently by Ceresole et al. in [28].

We will present a Gorenstein canonical singularity Y whose horizon manifold is $Q^{1,1,1}$. Hence Y can be regarded as a limiting space of Calabi-Yau manifolds. We will also find interesting relations with theories involving M2 branes placed at an orbifold singularity $\mathbf{S}^5/\mathbb{Z}_2 \times \mathbb{Z}_2$. The homogeneous space $Q^{1,1,1} = (\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2))/(\mathrm{U}(1) \times \mathrm{U}(1))$ may be obtained by blowing up the orbifold $\mathbf{S}^5/\mathbb{Z}_2 \times \mathbb{Z}_2$. This orbifold can be seen as two D6 branes at an $\mathbb{R}^4/\mathbb{Z}_2$ orbifold or as an elliptic model with D3 branes on a circle together with two NS5 branes and two D5 branes.

In section 2 we begin with a brief introduction to Sasaki-Einstein manifolds and provide a Calabi-Yau Gorenstein canonical singularity Y whose horizon is the Sasaki-Einstein manifold $Q^{1,1,1}$. In section 3 we show that topologically, $Q^{1,1,1}$ is the same as the complex blow-up of the orbifold $\mathbf{S}^7/\mathbb{Z}_2 \times \mathbb{Z}_2$. In section 4 we discuss the field theory obtained on the brane world-volume placed on the Calabi-Yau Gorenstein canonical singularity Y.

2. Near-horizon geometries of cone branes

Let (X, g_X) be a Riemannian manifold of real dimension $2n-1$ and $\mathbb{R}_+$ be the open half-line $0 < r < \infty$. Then the metric cone C(X) over X is a Riemannian manifold $\mathbb{R}_+ \times X$ with a metric

$$g_{C(X)} = dr^2 + r^2 g_X \, . \tag{2.1}$$

We often add one point (which is called the vertex) to $C(X)$ corresponding to the location $r = 0$ and use the same notation $C(X)$.

From the relation between Ricci curvature of X and that of its metric cone $C(X)$, we can show that X is Einstein with positive curvature if and only if $C(X)$ is Ricci-flat [36, 21]. However $C(X)$ will be metrically singular at the vertex of the cone except when X is a sphere, in which case $C(X)$ is actually flat.

We say that (X, g_X) is *Sasakian* if the holonomy group of the metric cone $(C(X), g_{C(X)})$ reduces to a subgroup of $\mathrm{U}(n)$. In particular, $(C(X), g_{C(X)})$ is Kähler. Therefore, (X, g_X) is Sasaki-Einstein if and only if its metric cone $C(X)$ is Calabi-Yau (Kähler Ricci-flat).

Let I be a parallel complex structure on $C(X)$, i.e. I commutes with the Levi-Civita connection on $C(X)$. Then $\chi := I(\partial r)$ will be a unit Killing vector field on X where X is identified with $X \times \{1\}$. The Killing vector field χ on X defines a foliation $\mathcal{F}$ whose leaves are the integral curves of χ. X is called a *regular Sasakian* manifold when these leaves are closed and have the same length. In this case, χ defines a U(1) action on X and X can be understood as a circle bundle over the $(2d-2)$ real dimensional manifold M, which is the space of the leaves. The CR structure on X pushes down to give a complex structure on M and the Sasakian condition on X will guarantee that the complex structure will be Kähler. By a version of the Kodaira Embedding Theorem [37], M will be a projective variety. Thus $C(X)$ is a $\mathbf{C}^*$-bundle over a projective variety M and this can be realized as an affine cone over M (which will be described in more detail for a Sasaki-Einstein 7-manifold $Q^{1,1,1}$ later). It also can be shown that M is a $\mathbf{Q}$-factorial Fano variety. For details, we refer the reader to [34].

The above process can be reversed in the following sense. Consider a complex variety $Y \subset \mathbb{C}^n$ of complex dimension d such that $0 \in Y$. Let $\rho : \mathbb{C}^n \to \mathbb{R}$ be the square of the usual distance function, namely

$$\rho(y) = |y_1|^2 + \cdots + |y_n|^2 \tag{2.2}$$

for $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$. It is easy to see that there exists an $\epsilon_0 > 0$ such that all the $(2d-1)$ real dimensional spheres $S_\epsilon = \rho^{-1}(\epsilon)$ for $\epsilon_0 \geq \epsilon > 0$ are transversal to Y [35]. In particular, $S_\epsilon \cap Y$ is smooth and has the equivalent Riemannian structure for all $\epsilon \in (0, \epsilon_0)$. We denote it by $L(Y, 0)$ and call it the *link* (or horizon) of Y at 0. In fact, integration along the Euler vector fields $\rho\partial/\partial\rho$ produces an isometry

$$\phi : L(Y, 0) \times (0, \epsilon_0) \to (B_{\epsilon_0} \setminus \{0\}) \cap Y \tag{2.3}$$

such that $\rho(\phi(x, t)) = t$ where $B_{\epsilon_0} = \{x \in \mathbb{C}^n : \rho(x) < \epsilon_0\}$. Hence $(B_{\epsilon_0} \setminus \{0\}) \cap Y$ is the metric cone over the link $L(Y, 0)$. The link (or horizon) $L(Y, 0)$ is Sasaki-Einstein if and only if the singularity $(Y, 0)$ is Calabi-Yau. Let I be the complex structure on Y. If $L(Y, 0)$ is regular, then the Killing vector field $\chi := I(\rho\frac{\partial}{\partial\rho})$ defines a foliation whose leaves are circles of the same length. This implies that the $\mathbf{C}^*$-action on $\mathbb{C}^n$ induces an action on Y and M is its quotient. Thus M is a projective variety and Y is an affine cone over M.

In this paper, we will consider a regular Sasaki-Einstein 7-manifold $Q^{1,1,1}$ which is known to be a U(1) bundle over a threefold $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1$ with winding number $(1, 1, 1)$. Recall [27] that $Q^{p,q,r}$ is a homogeneous space $(\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2))/(\mathrm{U}(1) \times \mathrm{U}(1))$, where $\mathrm{U}(1) \times \mathrm{U}(1)$ is embedded as follows: Consider an embedding of $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$ as the standard maximal torus into $(\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2))$. Thus each U(1) is embedded into SU(2) via the third Pauli matrix on the Lie algebra level. Let $\mathbf{h} = \mathbf{u(1)} \oplus \mathbf{u(1)}$ be the Lie subalgebra orthogonal to the Lie subalgebra generated by $p\sigma_{z,1} + q\sigma_{z,2} + r\sigma_{z,3}$ in the Lie algebra $\mathbf{u(1)} \oplus \mathbf{u(1)} \oplus \mathbf{u(1)}$ where $\sigma_{z,i}$ are the generators of $\mathbf{u(1)}$. Then we embed $\mathbf{h}$ into $\mathbf{su(2)} \oplus \mathbf{su(2)} \oplus \mathbf{su(2)}$ by the embedding described above. Its quotient in the Lie group level is denoted by $Q^{p,q,r}$. By taking the further quotient of $Q^{p,q,r}$ by the Lie subalgebra generated by $p\sigma_{z,1} + q\sigma_{z,2} + r\sigma_{z,3}$, one can see that $Q^{p,q,r}$ is a U(1)-bundle over $(\mathrm{SU}(2)/\mathrm{U}(1)) \times (\mathrm{SU}(2)/\mathrm{U}(1)) \times (\mathrm{SU}(2))/\mathrm{U}(1))$ which is $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1$. The complexification of this U(1)-bundle over $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1$ will be $\mathcal{O}_{\mathbf{CP}^1}(-p) \otimes \mathcal{O}_{\mathbf{CP}^1}(-q) \otimes \mathcal{O}_{\mathbf{CP}^1}(-r)$. If we embed $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1$ by $\mathcal{O}_{\mathbf{CP}^1}(p) \otimes \mathcal{O}_{\mathbf{CP}^1}(q) \otimes \mathcal{O}_{\mathbf{CP}^1}(r)$ into a projective space $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{CP}^1}(p) \otimes \mathcal{O}_{\mathbf{CP}^1}(q) \otimes \mathcal{O}_{\mathbf{CP}^1}(r)))$ and take the affine cone over the image, the vertex will be singular and the exceptional divisor over the vertex in the blowing-up will be $\mathcal{O}_{\mathbf{CP}^1}(-p) \otimes \mathcal{O}_{\mathbf{CP}^1}(-q) \otimes \mathcal{O}_{\mathbf{CP}^1}(-r)$.

We will now provide more details about $Q^{1,1,1}$. Consider the following embedding

$$\sigma : \mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1 \hookrightarrow \mathbf{CP}^7 \tag{2.4}$$

by a very ample line bundle $\mathcal{O}_{\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1}(1,1,1)$, which can be expressed as a map

$$(s:t) \times (u:v) \times (w:x) \to (suw : sux : svw : svx : tuw : tux : tvw : tvx) \tag{2.5}$$

sending the tri-homogeneous coordinates of $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1$ into the homogeneous coordinates of $\mathbf{CP}^7$. We denote the image by

$$Z := \sigma(\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1)\,. \tag{2.6}$$

Let $q : \mathbb{C}^8 \setminus \{0\} \to \mathbf{CP}^7$ be the natural quotient map of $\mathbb{C}^8 \setminus \{0\}$ by the $\mathbf{C}^*$-action. Then the affine cone Y over Z

$$Y := q^{-1}(Z) \cup \{0\} \tag{2.7}$$

is the desired singularity of a Calabi-Yau fourfold. Thus Y is the zero locus of the ideal I generated by the kernel of the map

$$\begin{aligned} &\sigma^* : \mathbf{C}[z_0, z_1, \ldots, z_7] \longrightarrow \mathbf{C}[s,t,u,v,w,x]\,, \\ &\quad \sigma^*(z_0) = suw\,, \quad \sigma^*(z_1) = sux\,, \quad \sigma^*(z_2) = svw\,, \quad \sigma^*(z_3) = svx\,, \\ &\quad \sigma^*(z_4) = tuw\,, \quad \sigma^*(z_5) = tux\,, \quad \sigma^*(z_6) = tvw\,, \quad \sigma^*(z_7) = tvx\,. \end{aligned} \tag{2.8}$$

One can compute the ideal I (for example, using the Gröbner basis) and hence one can show that Y is defined by the equations

$$\begin{aligned} z_0 z_3 - z_1 z_2 = z_0 z_5 - z_1 z_4 = z_0 z_6 - z_2 z_4 = 0\,, \\ z_0 z_7 - z_1 z_6 = z_0 z_7 - z_2 z_5 = z_0 z_7 - z_3 z_4 = 0\,, \\ z_1 z_7 - z_3 z_5 = z_2 z_7 - z_3 z_6 = z_4 z_7 - z_5 z_6 = 0\,. \end{aligned} \tag{2.9}$$

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One can see that the ideal I can be generated by the quadrics from the following minimal free resolution of $\mathcal{O}_Z$:

$$0 \to \mathcal{O}_{\mathbf{P}^7}(-6) \to \bigoplus_9 \mathcal{O}_{\mathbf{P}^7}(-4) \to \bigoplus_{16} \mathcal{O}_{\mathbf{P}^7}(-3) \to \bigoplus_9 \mathcal{O}_{\mathbf{P}^7}(-2) \to \mathcal{O}_{\mathbf{P}^7} \to \mathcal{O}_Z \to 0\,. \tag{2.10}$$

It is easy to check all possible quadric generators for the ideal I. Now we want to show that Y has a Gorenstein canonical singularity. This is the requirement that Y be at finite distance with respect to the natural Weil-Peterson metric on the moduli of Calabi-Yau manifolds [32, 39, 40]. Since Z is projectively normal, Y is normal. Thus Y is toric because there is a $\mathrm{U}(1)^4$-action on Y. (You may also explicitly compute the toric fan Δ which realizes this toric embedding. The fan Δ is nothing but a dual cone σ in $\mathbb{Z}^6$ to the convex rational polyhedral cone $\check{\sigma}$ generated by $(1,0,1,0,1,0)$, $(1,0,1,0,0,1)$, $(1,0,0,1,1,0)$, $(1,0,0,1,0,1)$, $(0,1,1,0,1,0)$, $(0,1,1,0,0,1)$, $(0,1,0,1,1,0)$, $(0,1,0,1,0,1)$, corresponding to the monomials which appear in (2.8). We refer to [41, 42] for a review of toric geometry.) Since the vertex of Y has a rational singularity (because it is toric), Y has a canonical singularity when it is Gorenstein [38]. One can prove that Y is Gorenstein by computing the dualizing sheaf ω_Y of Y (which is the same as the bundle of holomorphic 4-forms on the smooth part of Y) using the fact that Z is subcanonical. Note that since Y is an affine toric variety, the dualizing sheaf ω_Y is in fact trivial, which means that there is a non-vanishing holomorphic 4-form on Y. (You can also show this fact by a direct computation of ω_Y using the resolution (2.10).) As stated in [32], this means that there exists a non-vanishing holomorphic 4-form on Y which extends to a holomorphic 4-form on any smooth resolution of Y.

In this setting, $Q^{1,1,1} = L(Y,0)$. Thus $Q^{1,1,1}$ can be taken to be the intersection of Y with the unit sphere in $\mathbb{C}^8$:

$$|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 + |z_6|^2 + |z_7|^2 = 1\,. \tag{2.11}$$

The equations for Y can be 'solved' by setting

$$\begin{array}{llll} z_0 = SC_1D_1\,, & z_1 = SC_1D_2\,, & z_2 = SC_2D_1\,, & z_3 = SC_2D_2\,, \\ z_4 = TC_1D_1\,, & z_5 = TC_1D_2\,, & z_6 = TC_2D_1\,, & z_7 = TC_2D_2\,. \end{array} \tag{2.12}$$

where (S,T), (C_1,C_2), (D_1,D_2) are three pairs of homogeneous coordinates for the three $\mathbf{CP}^1$ spaces. It is easy to check that this choice for z_i satisfies equations (2.9).

If we write now $S = A_2B_1$ and $T = A_1B_2$, they are invariant under

$$(A_1,A_2) \to \lambda(A_1,A_2)\,, \qquad (B_1,B_2) \to \lambda^{-1}(B_1,B_2)\,. \tag{2.13}$$

The z_i are invariant under

$$(C_1,C_2) \to \gamma(C_1,C_2)\,, \qquad (D_1,D_2) \to \gamma^{-1}(D_1,D_2)\,, \tag{2.14}$$

where λ and γ are two complex numbers.

We can now choose the real parts of (possibly different for each of the following equations) of λ and γ to set:

$$|B_1|^2 + |C_1|^2 + |C_2|^2 = |A_1|^2 + |D_1|^2 + |D_2|^2 \tag{2.15}$$

and

$$|B_2|^2 + |C_1|^2 + |C_2|^2 = |A_2|^2 + |D_1|^2 + |D_2|^2 \,. \tag{2.16}$$

To identify the manifold $Q^{1,1,1}$, we set $|C_1|^2+|C_2|^2 = |D_1|^2+|D_2|^2 = |S|^2+|T|^2 = 1$. Hence the real isometry group is $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$. We now divide this by the angular parts of (2.13) and (2.14) which gives two $\mathrm{U}(1)$ groups. Finally we obtain $Q^{1,1,1} = (\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2))/(\mathrm{U}(1) \times \mathrm{U}(1))$.

Our goal is to find the $\mathcal{N} = 2$ superconformal theory which is dual to the M theory compactified on $AdS_4 \times Q^{1,1,1}$, in the infrared limit of the field theory of N coincident M2 branes placed at a conifold singularity.

3. Comparison to an $\mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{R}^4/\mathbb{Z}_2$ orbifold.

In order to set up our proposed duality, we need to compare the conifold to a $\mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{R}^4/\mathbb{Z}_2$ orbifold background. Consider an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{R}^8$:

$$(1,0) \cdot (x_1, \ldots, x_8) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8) \,, \tag{3.1}$$

$$(0,1) \cdot (x_1, \ldots, x_8) = (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8) \,, \tag{3.2}$$

where $(1,0)$ (resp. $(0,1)$) is the generator of the first (resp. second) factor of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $x_1, \ldots, x_n$ are the coordinates of $\mathbb{R}^8$, and an action of $\mathbb{Z}_2$ on $\mathbb{R}^4$ via a multiplication by -1.

Recall that the complex blow-up of $\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{Z}_2$ is a blow-up of the quotient space of $\mathbb{C}^4$ as a complex space after identification of $\mathbb{C}^4$ with $\mathbb{R}^8$. First note that there is an isomorphism between the orbifold spaces

$$\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{R}^4/\mathbb{Z}_2 \,. \tag{3.3}$$

Let

$$\pi : Bl(\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{Z}_2) \to \mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{Z}_2 \,, \tag{3.4}$$

$$\pi' : Bl(\mathbb{R}^4/\mathbb{Z}_2) \to \mathbb{R}^4/\mathbb{Z}_2 \tag{3.5}$$

be the complex blow-ups of the orbifold singularities $\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{R}^4/\mathbb{Z}_2$ respectively. By (3.3), we have

$$Bl(\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{Z}_2) \cong Bl(\mathbb{R}^4/\mathbb{Z}_2) \times Bl(\mathbb{R}^4/\mathbb{Z}_2) \,. \tag{3.6}$$

The space $Bl(\mathbb{R}^4/\mathbb{Z}_2)$ can be regarded as the total space of a line bundle $\mathcal{O}_{\mathbf{CP}^1}(-1)$. Hence it is a complex line bundle over $\mathbf{S}^2$. Therefore there is a vector bundle map

$$q : Bl(\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{Z}_2) \longrightarrow \mathbf{S}^2 \times \mathbf{S}^2 \,, \tag{3.7}$$

with fibers $\mathbb{R}^4$. We define the complex blow-up of the orbifold singularity $\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2$ via the map π in (3.4):

$$Bl(\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2) = \pi^{-1}(\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2). \tag{3.8}$$

Thus we have the following diagram:

$$\begin{tikzcd}
Bl(\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2) \arrow[r, hook, "i_{Bl}"] \arrow[d] & Bl(\mathbb{R}^8/\mathbb{Z}_2\times\mathbb{Z}_2) \arrow[d] \\
\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2 \arrow[r, hook, "i"] & \mathbb{R}^8/\mathbb{Z}_2\times\mathbb{Z}_2.
\end{tikzcd}$$

Now we study the complex blow-up $Bl(\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2)$. There is a map

$$q_S : Bl(\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2) \to \mathbf{S}^2\times\mathbf{S}^2 \tag{3.9}$$

which is a composition of the natural inclusion i_{Bl} and the map q in (3.7). Since i_{Bl} and q are transversal, the map q_S in (3.9) is a smooth fibration and it is easy to see the fiber is $\mathbf{S}^3$. In a summary, we have the following fibrations:

$$\begin{tikzcd}
 & Bl(\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2) \arrow[r, hook, "i_{Bl}"] & Bl(\mathbb{R}^8/\mathbb{Z}_2\times\mathbb{Z}_2) \arrow[dd, "q"] \\
\mathbf{S}^3 \arrow[ur] \arrow[r, hook] & \mathbb{R}^4 \arrow[ur] & \\
 & & \mathbf{S}^2\times\mathbf{S}^2
\end{tikzcd}$$

Thus $Bl(\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2)$ is an $\mathbf{S}^3$ bundle over $\mathbf{S}^2\times\mathbf{S}^2$. Now we claim that both $Q^{1,1,1}$ and $Bl(\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2)$ are topologically trivial $\mathbf{S}^3$ bundles over $\mathbf{S}^2\times\mathbf{S}^2$. In order to show that $Bl(\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2)$ is a topologically trivial $\mathbf{S}^3$ bundle, note that the blow-up $Bl(\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2)$ can be achieved in two steps. First we consider a blow-up $Bl(\mathbf{S}^7/\mathbb{Z}_2)$, where $\mathbb{Z}_2$ acts only on the first four coordinates by -1. Then the situation is the same as in [21] except we have two more extra coordinates. Thus the blow-up $Bl(\mathbf{S}^7/\mathbb{Z}_2)$ will be a trivial $\mathbf{S}^5$ bundle over $\mathbf{S}^2$. Now we further blow up the space $Bl(\mathbf{S}^7/\mathbb{Z}_2)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on the last four coordinates. This will act on the fiber $\mathbf{S}^5$ and the blow-up of $Bl(\mathbf{S}^7/\mathbb{Z}_2)/\mathbb{Z}_2$ will be the trivial bundle over $\mathbf{S}^2$. Since the space $Bl(\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2)$ is the same as the space obtained by the two successive of blow-ups we described, we can conclude that $Bl(\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2)$ is topologically $\mathbf{S}^3\times\mathbf{S}^2\times\mathbf{S}^2$. In the case of $Q^{1,1,1} = (\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2))/(\mathrm{U}(1)\times\mathrm{U}(1))$, the triviality can be proved by observing that the 'forgetting' map of the last SU(2) will give a fibration of $\mathbf{S}^3$ over $\mathbf{S}^2\times\mathbf{S}^2$, which is trivial.

4. $\mathcal{N}=2$ field theory in 3 dimensions

In this section we will build the field theory corresponding to the M theory compactified on the conifold Y. In order to do this we will use the above identification[1] of $Q^{1,1,1}$ with the blow-up of the orbifold $Bl(\mathbf{S}^7/\mathbb{Z}_2\times\mathbb{Z}_2)$ as $\mathbf{S}^3$ bundles over $\mathbf{S}^2\times\mathbf{S}^2$.

4.1 Orbifold theory

We begin by briefly reviewing the paper by Klebanov and Witten [21]. They considered $AdS_5\times\mathbf{S}^5/\mathbb{Z}_2$ background where $\mathbb{Z}_2$ acts on four of the six coordinates of $\mathbf{S^5}$. In terms of branes, this corresponds to D3 branes on a transversal $\mathbb{C}^2/\mathbb{Z}_2\times\mathbb{C}$ space which, after taking a T-duality on one of the directions of $\mathbb{C}^2/\mathbb{Z}_2$, gives an elliptic model with D4 branes between NS5 branes. The gauge group is $\mathrm{SU}(N)\times\mathrm{SU}(N)$ and at each NS5 brane we have an $(N,\bar{N})$ field. The rotation of NS5 branes corresponds to adding a mass for the adjoint field, and results in an $\mathcal{N}=1$ theory with two copies of $(N,\bar{N})\oplus(\bar{N},N)$. There is an action of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ group on the two copies of $(N,\bar{N})\oplus(\bar{N},N)$ fields where the first factor SU(2) acts on two copies of $(N,\bar{N})$ and the second factor SU(2) acts on two copies of $(\bar{N},N)$.

In our case, where we have M2 branes on $\mathbb{C}^2/\mathbb{Z}_2\times\mathbb{C}^2/\mathbb{Z}_2$ space[2], we will use similar arguments. Consider type IIB string theory in ten dimension with two D6 branes placed on a product of two A_1 singularities. In eleven dimensions we begin with two types of KK monopoles, say in the (x^3,x^4,x^5,x^6) directions and (x^7,x^8,x^9,x^{10}) directions respectively, and M2 branes in the (x^1,x^2) directions. Reduction to 10 dimensions gives KK monopoles in the (x^3,x^4,x^5,x^6) directions, D6 branes in $(x^1,x^2,x^3,x^4,x^5,x^6)$ directions and D2 branes in the (x^1,x^2) directions. If the x^3 direction is compact, we make a T - duality with respect to it and we obtain a configuration with D3 branes on (x^1,x^2,x^3) directions, D5 branes in the (x^1,x^2,x^4,x^5,x^6) directions and NS5 branes in (x^1,x^2,x^7,x^8,x^9) directions.

We obtain an elliptic, self mirror model as an $\mathcal{N}=4$ supersymmetric field theory with a gauge group $\mathrm{U}(N)\times\mathrm{U}(N)$ with matter given by A_1 in $(\bar{N},1)$, B_1 in $(N,1)$, A_2 in $(1,\bar{N})$, B_2 in $(1,N)$, C_1,C_2 in $(N,\bar{N})$ and D_1,D_2 in the $(\bar{N},N)$ (we discuss the fields in the $\mathcal{N}=2$ language). The model also contains fields in the adjoint representation of the two U(N) groups, denoted by Φ and $\tilde{\Phi}$. The A_i,B_i fields are given by strings with one end on D3 and the other end on either of the two D5 branes. The C_i,D_i fields are given by strings with both ends on the D3 branes

[1]The field theory discussion was suggested to us by Matthew Strassler and we refer to [33] for a detailed and interesting discussion about exactly marginal operators in three dimensions.

[2]The choice for an orbifold was suggested by Matthew Strassler. He pointed out that it would give self mirror-symmetrical models where we could have a nice identification of marginal operators.

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stretched across either of the two NS branes. The superpotential is

$$g(A_1\Phi B_1 + A_2\tilde{\Phi}B_2) + g\mathrm{Tr}[\Phi(C_1D_1 + C_2D_2)] + g\mathrm{Tr}[\tilde{\Phi}(D_1C_1 + D_2C_2)]\,. \qquad (4.1)$$

To arrive to an $\mathcal{N} = 2$ theory, we add a mass term for the Φ and $\tilde{\Phi}$. This corresponds to rotating the NS branes in the (x^5, x^6, x^8, x^9) directions. But the rotation of the D5 brane induces a change in the coupling between the hyper-multiplets A and B and the adjoint multiplets. In order to preserve the self-duality, we need to rotate the NS5 branes and the D5 branes by the same angle. In other words, we need to add the following to the superpotential:

$$\frac{m}{2}\Big(\mathrm{Tr}(\Phi^2 + \tilde{\Phi}^2) + \mathrm{Tr}(A_1B_1 - A_2B_2)^2 + \mathrm{Tr}(C_1D_1 - C_2D_2)^2\Big)\,. \qquad (4.2)$$

In equation (4.2), m is the mass of the adjoint field or the tangent of the rotation angle by which the D5 branes are rotated. If we add all the above discussed terms and integrate out the adjoint field Φ, the following superpotential is obtained:

$$\frac{g^2}{2m}\Big[\epsilon^{ij}\epsilon^{kl}\mathrm{Tr}(A_iB_kA_jB_l) + \epsilon^{ij}\epsilon^{mn}\mathrm{Tr}(C_iD_mC_jD_n)\Big]\,. \qquad (4.3)$$

This is the term obtained by rotating all the branes by the same angle and by integrating out the massive fields (only the adjoint field acquires mass). Before rotation, the theory was $\mathcal{N} = 4$ and we could exchange the positions of the NS and D5 branes. After rotation, in order to do that, all the branes need to be rotated by the same angle. This implies that all the terms in (4.2) are needed in order to recover a self-dual theory. As explained in [33] we actually require that the $\mathbb{Z}_2$ symmetry exchanging the two $\mathbb{Z}_2$ orbifolds is preserved. The $\mathbb{Z}_2$ symmetry determines a marginal deformation from one self-dual model to another. The theory has a line of fixed points which contains the $\mathcal{N} = 4$ theory.

The field theory obtained by taking M theory on $\mathbf{S}^7/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is, in our case, equivalent to the $\mathcal{N} = 2$ three dimensional theory obtained by adding the superpotential (4.3) to the $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge theory with the matter content given by the fields A, B, C, D.

We want to make an important observation here. The brane configuration described is invariant under an S-duality which replaces the NS branes by D5 branes and the D5 branes by NS branes, and leaves the D3 branes invariant. This S-duality replaces the A and B fields by the C and D fields and vice-versa. In equation (4.1), C_i and D_i appear in the first term and A_i and B_i appear in the second term. Equation (4.2) remains invariant under exchanging of A, B and C, D fields. By remembering that the two U(1) groups of the conifold act on pairs A, B and C, D, the S-duality just inverts these two U(1) groups. One of the SU(2) groups will act upon (A_1, A_2), one on (B_1, B_2) and the third one on $S' = C_2D_1, T' = C_1D_2$.

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4.2 Conifold theory

We now discuss the field theory obtained by starting from the conifold theory.

As in section 2, the parameterization of the conifold is given in terms of the fields S, T, C_i, D_i, where $S = A_2B_1$, $T = A_1B_2$. We consider a U(1) gauge theory with $\mathcal{N} = 2$ in three dimensions and introduce C_1, D_1 as chiral multiplets with charge 1 and C_2, D_2 as chiral multiplets with charge -1. The fields A_2, B_2 are neutral under this gauge group. We have the equation (2.15):

$$|B_1|^2 + |C_1|^2 + |C_2|^2 = |A_1|^2 + |D_1|^2 + |D_2|^2. \tag{4.4}$$

But this is just a D auxiliary field of the U(1) vector multiplet, by considering terms which do not involve the adjoint scalar obtained after reduction from $\mathcal{N} = 1, D = 4$. The moduli space of vacua is a part of the conifold. We consider now another U(1) gauge theory with $\mathcal{N} = 2$ in three dimensions. Let us introduce B_2, C_1, D_1 as chiral multiplets with charge 1 and C_2, D_2, A_2 with charge -1. The fields A_1, B_1 are neutral under this gauge group. We have the equation (2.16):

$$|B_2|^2 + |D_1|^2 + |D_2|^2 = |A_2|^2 + |C_1|^2 + |C_2|^2 \tag{4.5}$$

which can be again interpreted as a D auxiliary field of the U(1) vector multiplet. In the paper [21], one of the two gauge groups lives on the world-volume of their D3 branes. The conifold was identified with the moduli space of vacua of one of the U(1) groups. Here we do not have any U(1) group on the two-branes world-volume. So we need to identify the moduli spaces of both U(1) groups with the two branches of the conifold (2.15), (2.16). It is now clear why we need two equations for the conifold as compared with [21] where there was only one.

The result is that we have a theory with the gauge group $\mathrm{U}(1) \times \mathrm{U}(1)$ with chiral multiplets A_1, B_1, A_2, B_2 and C_1, C_2 and D_1, D_2 with charges $(-1, 0)$, $(1, 0)$, $(0, -1)$, $(0, 1)$, $(1, -1)$, $(-1, 1)$ respectively. The chiral multiplets describe the M2 brane motion on the conifold.

If we have N M2 branes the gauge theory is generalized to a $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge theory. Now the chiral fields C_i, D_i are matrices, the chiral fields A_i are row vectors and B_i are column vectors. It is then natural to use the S, T fields in order to have all the fields given by matrices. If all the matrices are diagonal, the diagonal entries of C_i, D_i give the positions of the N M2 branes at distinct points on the conifold. The gauge group is broken to a product of U(1) factors. But there are the extra-diagonal entries which need to be given masses in order to integrate out the unwanted massless chiral multiplets. We will do this by introducing a superpotential. It should preserve the symmetry of the conifold Y, i.e. $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)_R$ symmetry. The $\mathrm{U}(1)_R$ is the R-symmetry inherited by the $\mathcal{N} = 2$ three dimensional theory from its reduction from 4 dimensions. All the fields A_i, B_i, C_i, D_i have charge

1/2 and the fields S, T have charge 1. A superpotential which is invariant under $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ and has $\mathrm{U}(1)_R$ charge 2 can be written as:

$$W = \lambda\Big(\mathrm{Tr}(ST - TS) + \epsilon^{ij}\epsilon^{kl}\mathrm{Tr}(C_i D_k C_j D_l)\Big)\,, \tag{4.6}$$

where all the fields are now $N \times N$ matrices. If we now use the definition of S and T , the superpotential has a final form:

$$\frac{g^2}{2m}\Big[\epsilon^{ij}\epsilon^{kl}\mathrm{Tr}(A_i B_k A_j B_l) + \epsilon^{ij}\epsilon^{mn}\mathrm{Tr}(C_i D_m C_j D_n)\Big] \tag{4.7}$$

where the products $A_i B_k$ are to be understood as $N \times N$ matrices. This has charge 2 and is $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ invariant. The off-diagonal components receive mass from the superpotential (plus Higgs mechanism).

We observe that the above superpotential is the same as the one of (4.3). This is a marginal operator which takes us from a conformal theory to a new conformal field theory.

We now have the ingredients to state the result of this paper:

M theory on $AdS_4 \times Q^{1,1,1}$ is equivalent to the $N = 2$ three dimensional supersymmetric field theory obtained by starting with $N = 4$ three dimensional theory with the gauge group $\mathrm{U}(N) \times \mathrm{U}(N)$ and matter given by two fields in $(N, \bar{N}) \oplus (\bar{N}, N)$ and four fields in $(N, 1), (\bar{N}, 1), (1, N), (1, \bar{N})$ flowing to an infrared fixed point and then perturbed by the potential (4.3).

We end this section with a discussion of the conifold description of the configuration which is the S-dual of the one considered before. By remembering that the two U(1) groups of the conifold act on pairs A, B and C, D, the S-duality just inverts these two U(1) groups. One of the SU(2) groups will act upon (A_1, A_2), one on (B_1, B_2) and the third one on $S' = C_2 D_1$, $T' = C_1 D_2$.

In conclusion, in this paper we have extended the idea of comparing string/M theory at conifold singularities with superconformal field theories, to the case of $AdS_4 \times Q^{1,1,1}$ which is dual to superconformal field theory on three dimensions.

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